Shape Analysis Through Diffeomorphisms

Sylvain Arguillère
(CNRS, LPP)
Shape Analysis in Biology Workshop

Shape Analysis and Computational Anatomy

General idea:

- Shape analysis is the study of datasets of shapes, and their correlation with one another and other variables.

Shape Analysis and Computational Anatomy

General idea:

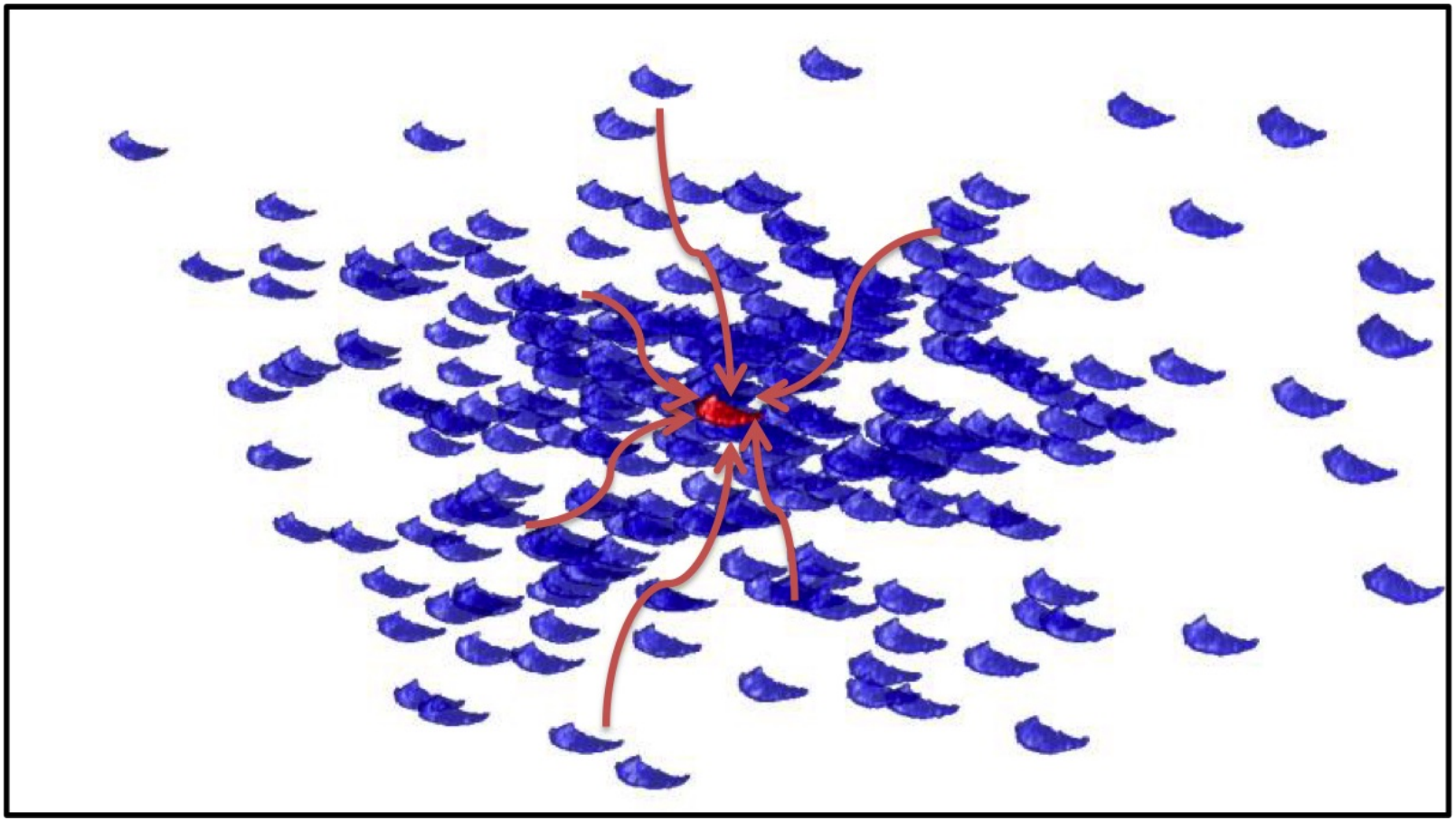
- Shape analysis is the study of datasets of shapes, and their correlation with one another and other variables.
- Build a suitable "shape space". Analyzing shapes involves:
  - **Moving in that space**, i.e., finding deformations along which shapes evolve from one instance to another,
  - **Comparing shapes** in the space, e.g., by finding deformations as above that requires the least "energy", so that bigger variations of shapes require higher energy.
  - **Parametrizing shape variations** around a given reference shape. This should allow the application of statistical methods on the data set that take into account the geometric variations between the shapes.

Shape Analysis and Computational Anatomy

General idea:

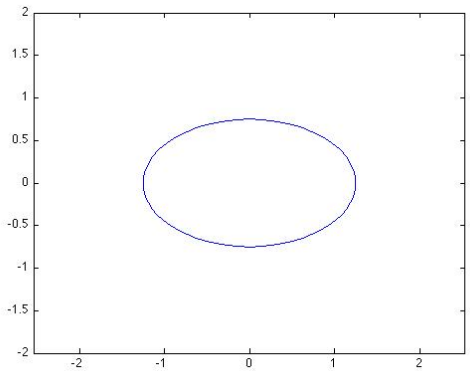
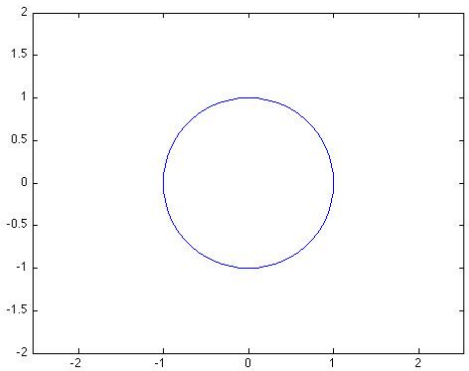
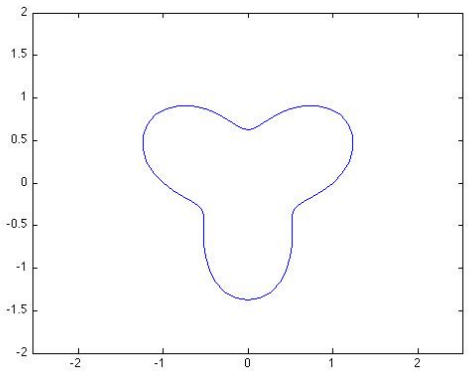
- Shape analysis is the study of datasets of shapes, and their correlation with one another and other variables.
- Build a suitable "shape space". Analyzing shapes involves:
  - **Moving in that space**, i.e., finding deformations along which shapes evolve from one instance to another,
  - **Comparing shapes** in the space, e.g., by finding deformations as above that requires the least "energy", so that bigger variations of shapes require higher energy.
  - **Parametrizing shape variations** around a given reference shape. This should allow the application of statistical methods on the data set that take into account the geometric variations between the shapes.
- Each of these steps requires some form of **shape registration**: finding a certain deformation from one shape onto another.

Illustration

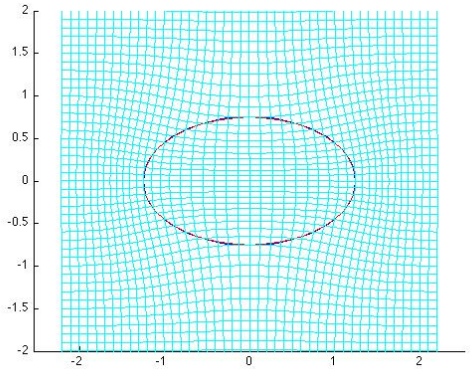
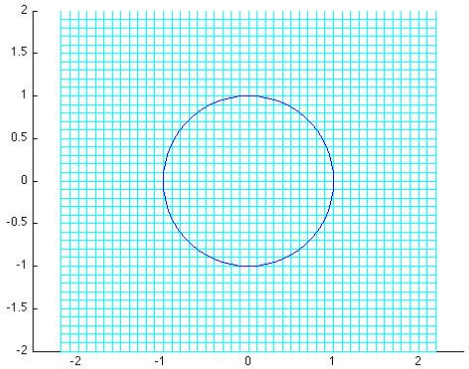
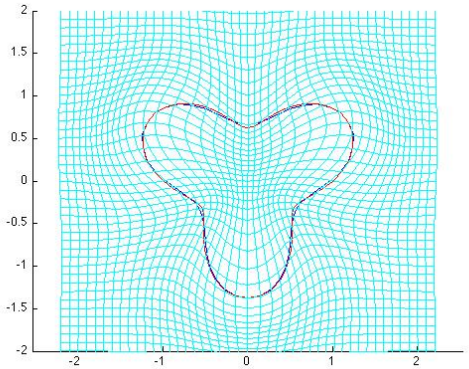


Shape registration

Goal: compare shapes while taking into account their geometric properties.

Idea: Use **diffeomorphisms**: deformations of the ambient space that preserve local and global geometric properties. The more different two shapes are, the more deformation is needed to map one close to the other.

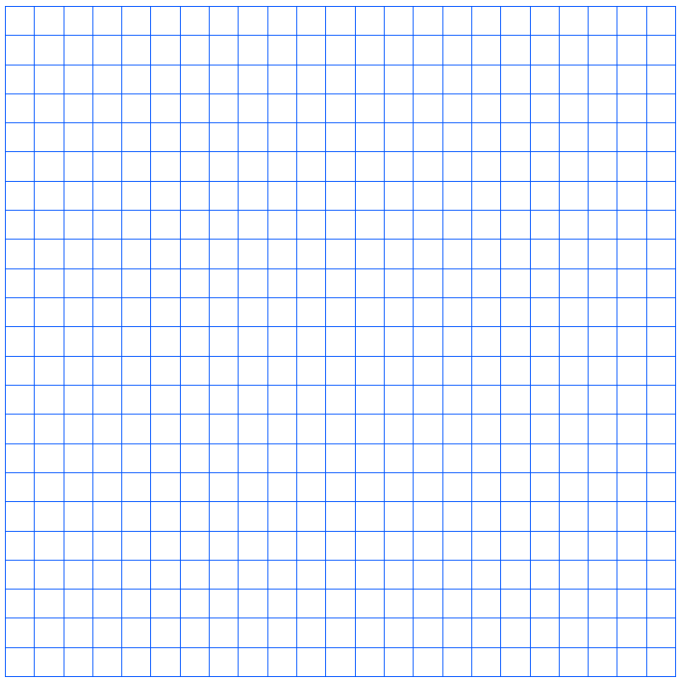




How to build diffeomorphisms

For $t \in [0,1]$, velocity field $v(t) : \mathbb{R}^d \to \mathbb{R}^d$. The position $x(t) \in \mathbb{R}^d$ at time t of a particle that moves along this velocity field is described by

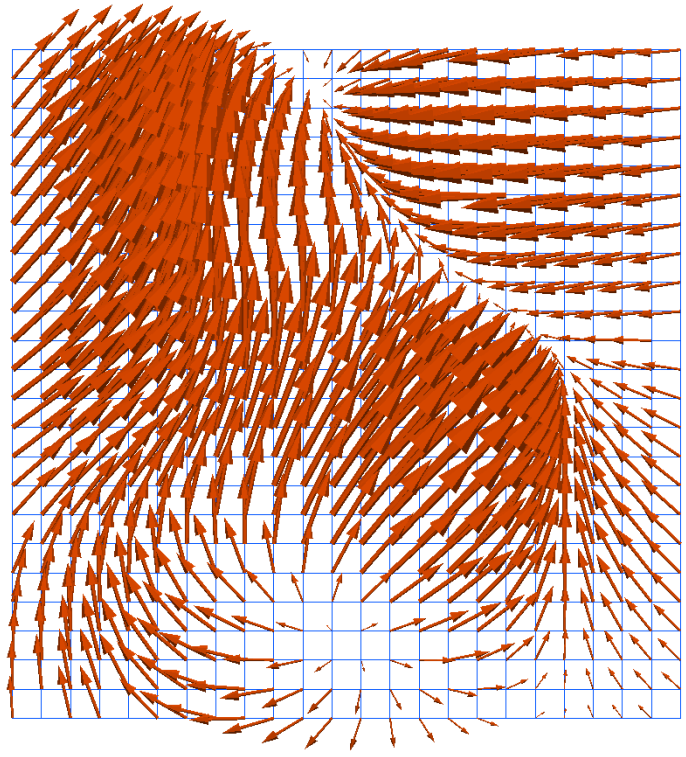
$$\frac{dx}{dt}(t) = v(t, x(t)).$$

Controlling Diffeomorphisms with Vector Fields



A controller specifies a direction at every point

Controlling Diffeomorphisms with Vector Fields



A controller specifies a direction at every point

How to build diffeomorphisms

For $t \in [0,1]$, velocity field $v(t) : \mathbb{R}^d \to \mathbb{R}^d$. The position $x(t) \in \mathbb{R}^d$ at time t of a particle that moves along this velocity field is described by

$$\frac{dx}{dt}(t) = v(t, x(t)).$$

This gives a deformation of the space at time t, denoted $\varphi(t)$, so that $\varphi(t, x)$ is the position at time t of the particle that started its motion at x at time 0. In particular, $\varphi(0, x) = x$.

How to build diffeomorphisms

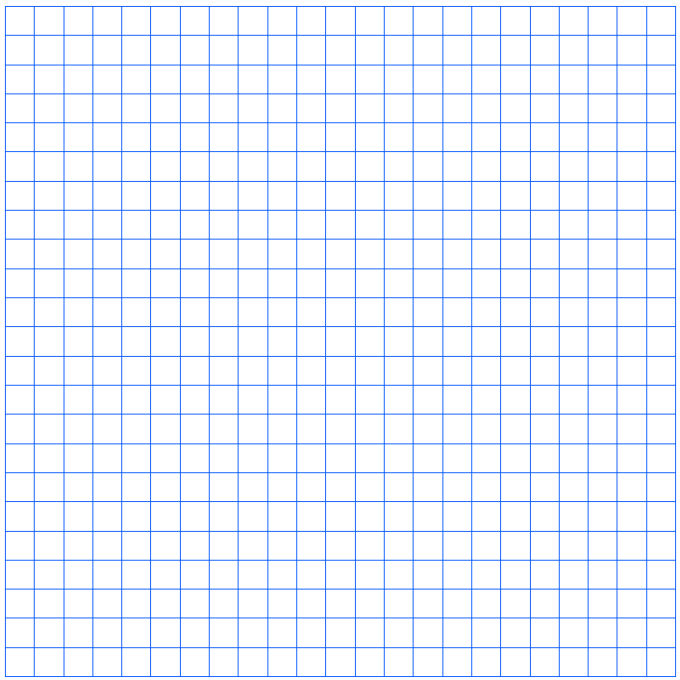
For $t \in [0,1]$, velocity field $v(t) : \mathbb{R}^d \to \mathbb{R}^d$. The position $x(t) \in \mathbb{R}^d$ at time t of a particle that moves along this velocity field is described by

$$\frac{dx}{dt}(t) = v(t, x(t)).$$

This gives a deformation of the space at time t, denoted $\varphi(t)$, so that $\varphi(t, x)$ is the position at time t of the particle that started its motion at x at time 0. In particular, $\varphi(0, x) = x$.

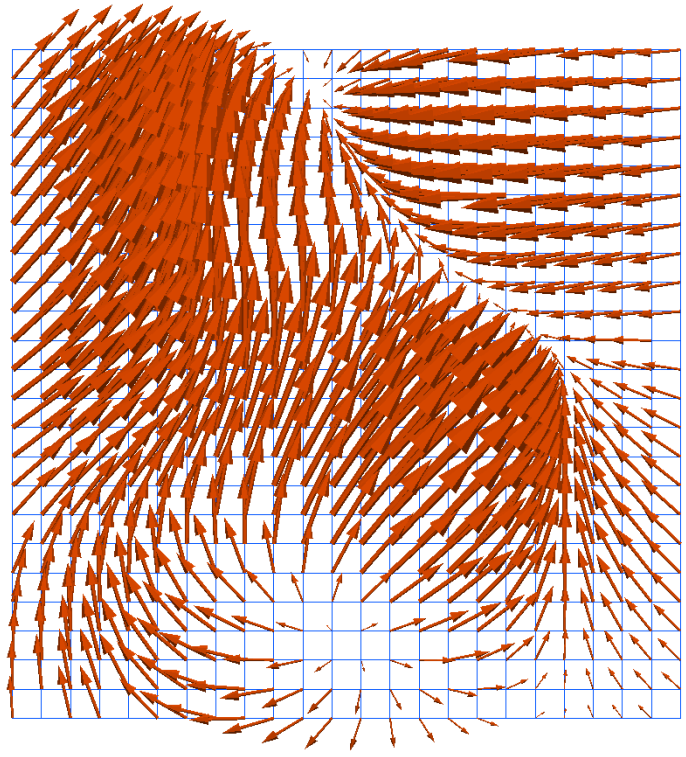
As long as we take $v(t)$ "very regular" with respect to the space variables, the transformation will be a **diffeomorphism**: it will map smooth curves onto smooth curves, corners onto corners, and preserve presence or lack of self-intersection points.

Controlling Diffeomorphisms with Vector Fields



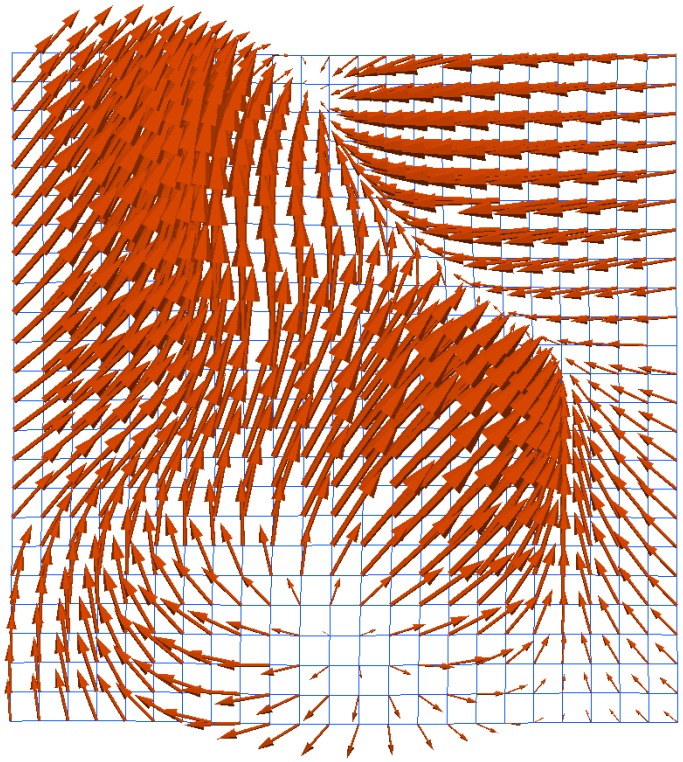
A controller specifies a direction at every point

Controlling Diffeomorphisms with Vector Fields



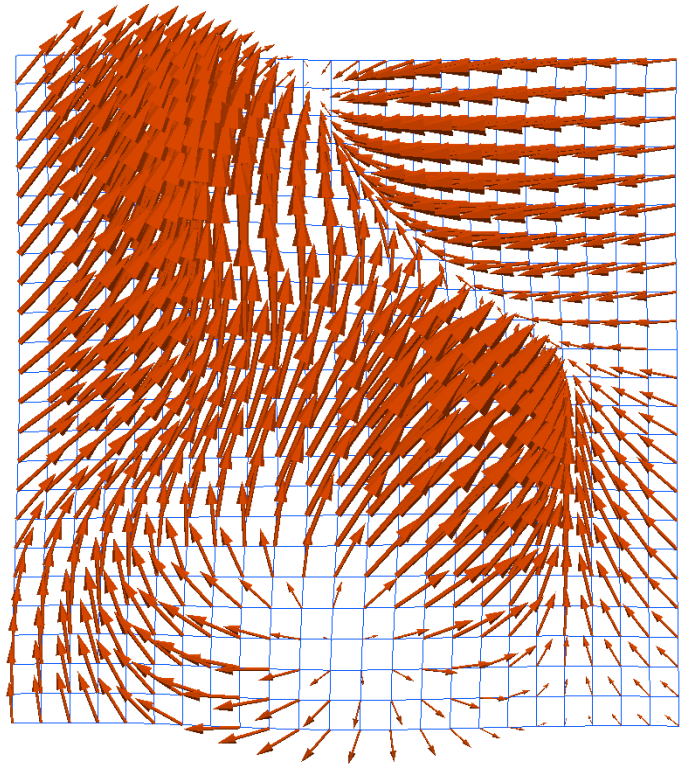
A controller specifies a direction at every point

Controlling Diffeomorphisms with Vector Fields



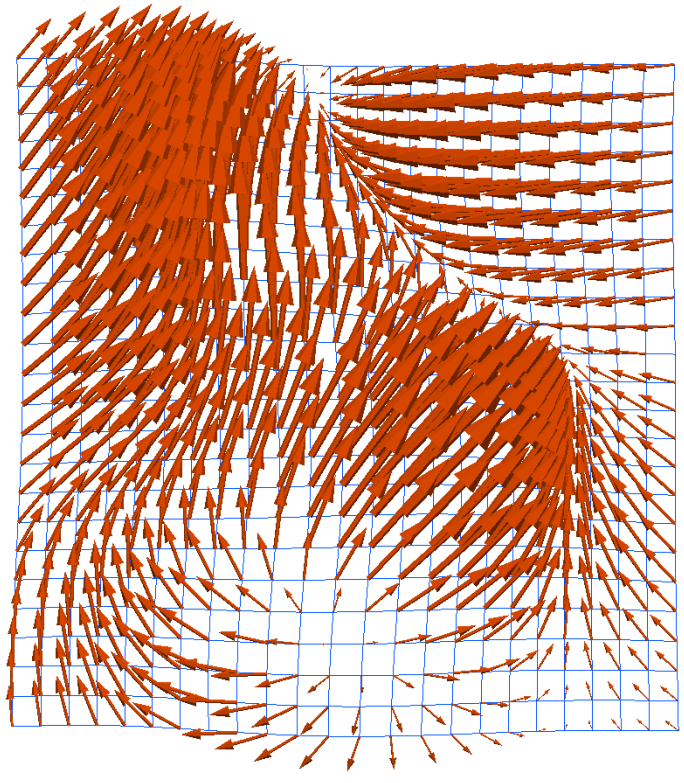
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



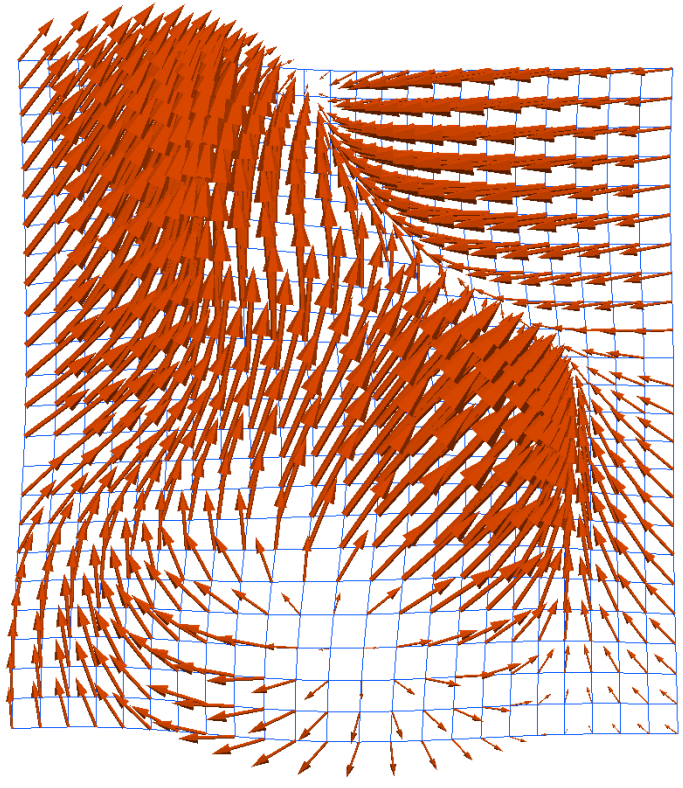
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



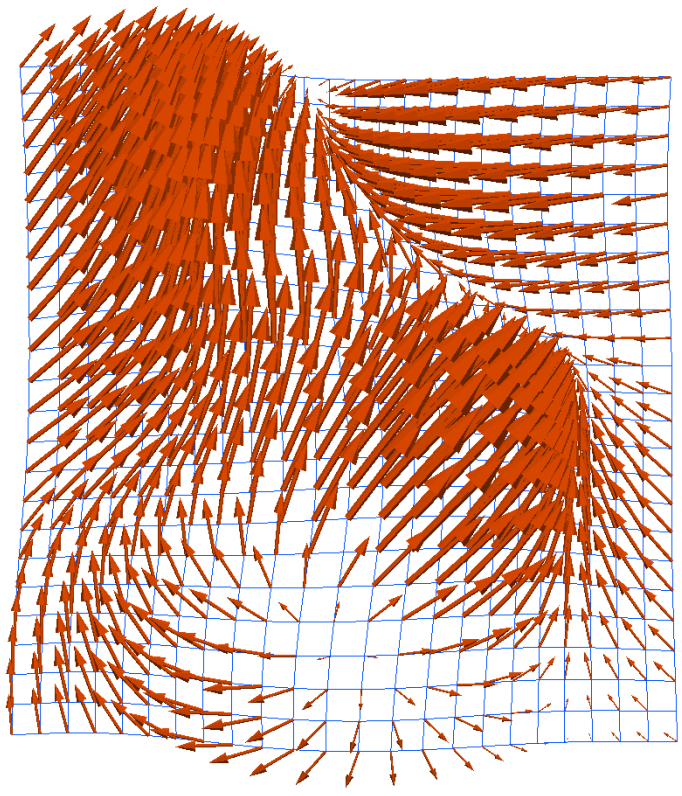
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



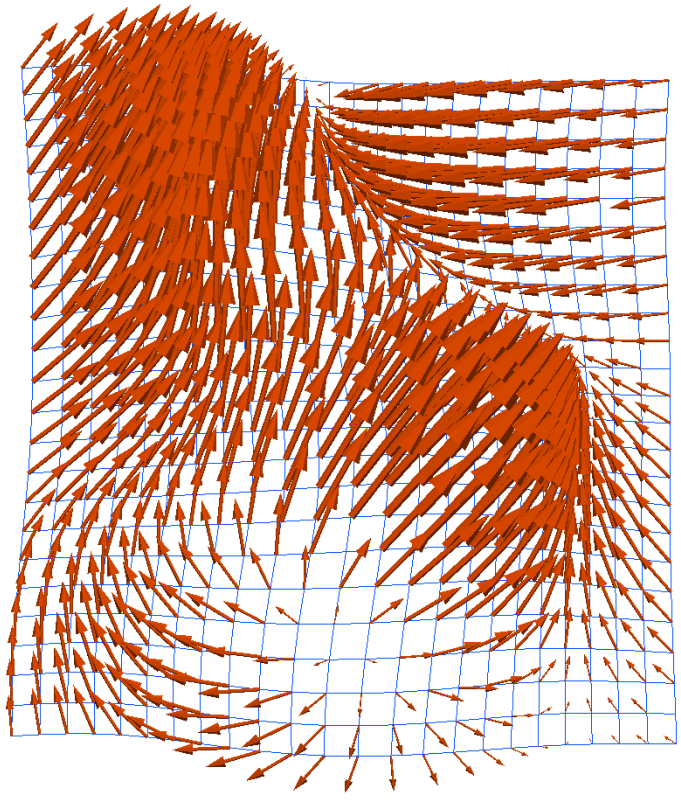
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



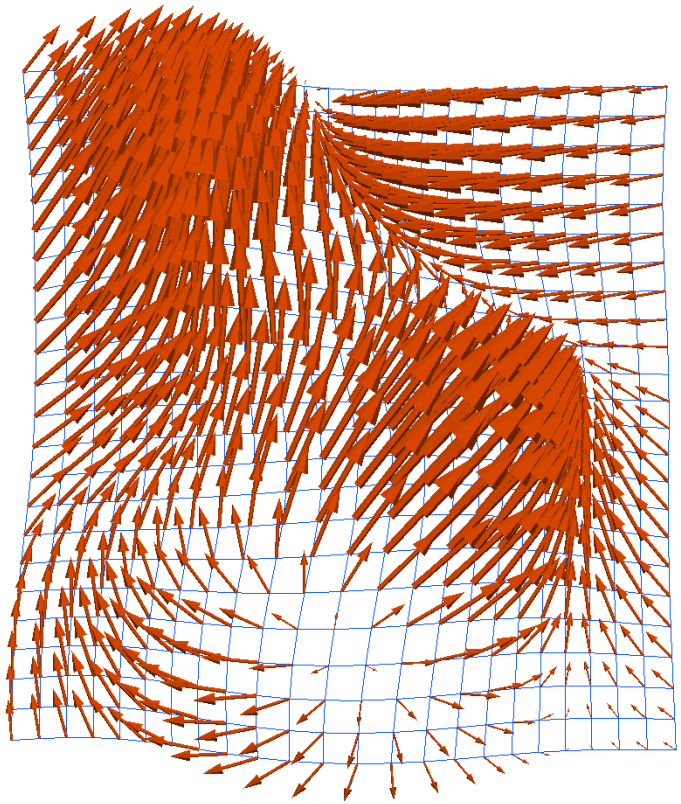
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



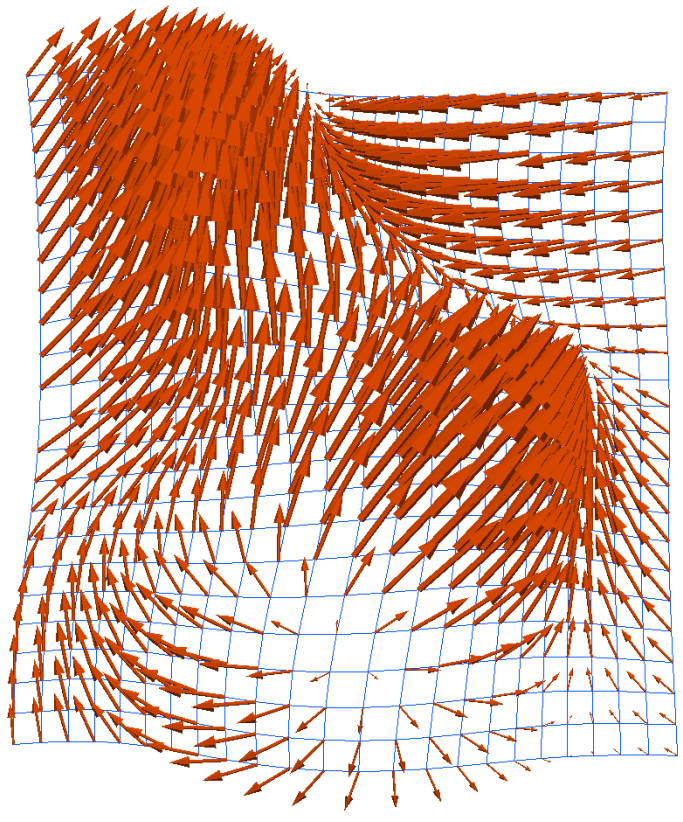
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



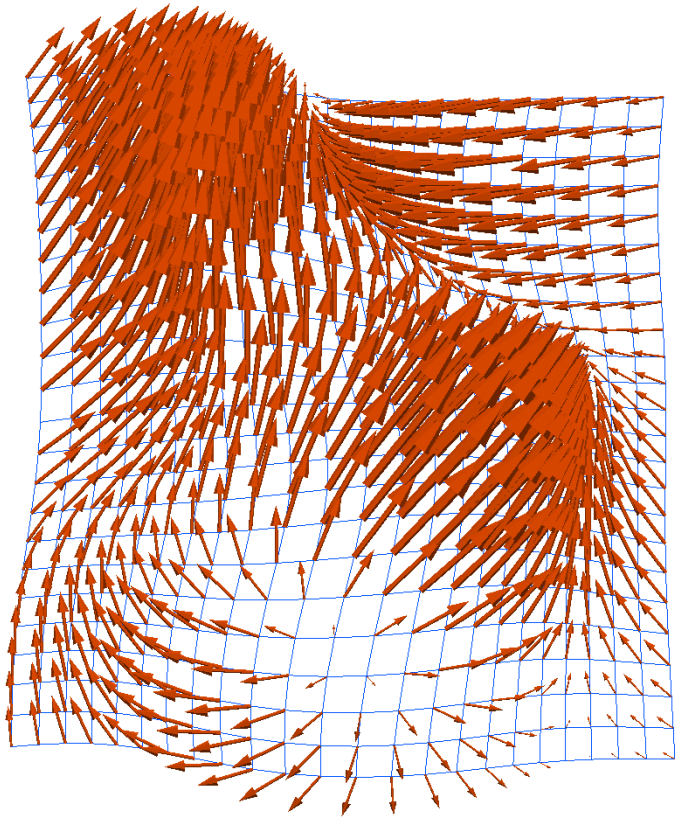
A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields

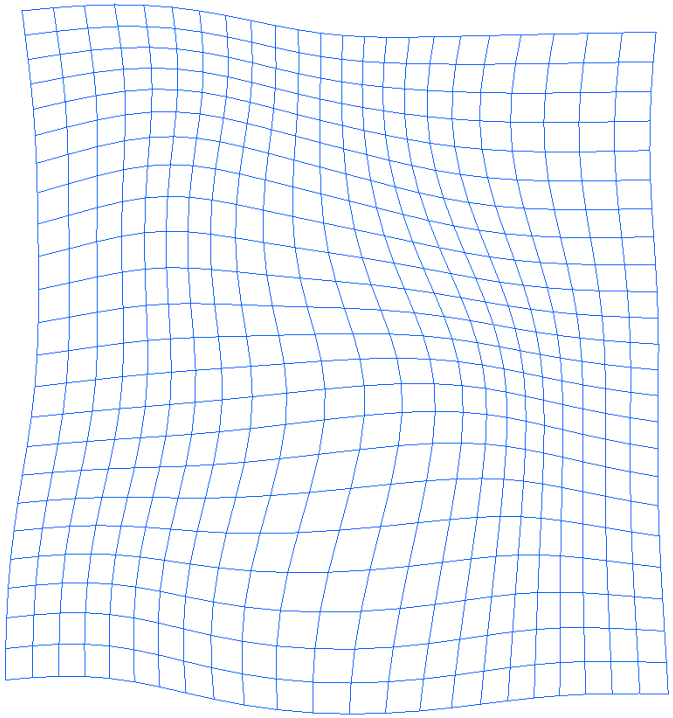


A controller specifies a direction at every point, at every time

Controlling Diffeomorphisms with Vector Fields



A controller specifies a direction at every point, at every time



Final Grid

Energy of Deformation for Shape Registration

- Fix a shape q_0, the **template**, from which we want to register another shape q_1 (the **target**).

Energy of Deformation for Shape Registration

- Fix a shape q_0, the **template**, from which we want to register another shape q_1 (the **target**).

- A time-dependent velocity field $(t, x) \mapsto v(t, x)$ yields a deformation $(t, x) \mapsto \varphi(t, x)$, which acts onto q_0 as denoted by $q(t) := \varphi(t) \cdot q_0$. The goal is now to find v^* which minimizes a functional

$$J(v) = \frac{1}{2} \int_0^1 \|v(t)\|_V^2 dt + g(q(1)),$$

where $\| \cdot \|_V$ is an appropriate Hilbert norm (for instance, one can take a sufficiently smooth Sobolev norm). The data atachment $g(q(1))$ is a crude measure of the difference between the deformed shape $q(1)$ and the target q_1.

Reduction

Proposition

For a certain appropriate $\|\cdot\|_V$, the following holds.

Assume the shape q_0 can be described by a finite family $q_0 = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$. For example, q_0 is a triangulated surface. Denote v^* a minimizer of the cost J.

Then for each time t, $v^*(t)$ is a sum of Gaussian vector fields centered at each x_i with fixed variance, that is,

$$v^*(t,x) = \sum_{i=1}^{n} p_i(t) e^{-\frac{|x - x_i(t)|^2}{\sigma^2}}, \quad p_i(t) \in \mathbb{R}^d, \sigma \in \mathbb{R}_+^*.$$

In this case,

$$\|v^*(t)\|_V^2 = \sum_{i,j=1}^{n} p_i(t)^T p_j(t) e^{-\frac{|x_i(t) - x_j(t)|^2}{\sigma^2}}.$$

We call $p(t)$ the **momentum** *of the deformation at time t.*

Remark: This finite dimensional reduction can be performed for more general Hilbert norms $\|\cdot\|_V$, although the formula for v^* would be slightly different.

Reduced Problem

We can simply work on $t \mapsto (p_1(t), \dots, p_n(t))$: we are left with minimizing

$$J_R(p_1, \dots, p_n) = \sum_{i,j=1}^{n} \int_0^1 p_i(t)^T p_j(t) e^{-\frac{|x_i(t)-x_j(t)|^2}{\sigma^2}} dt + g(x_1(1), \dots, x_n(1)),$$

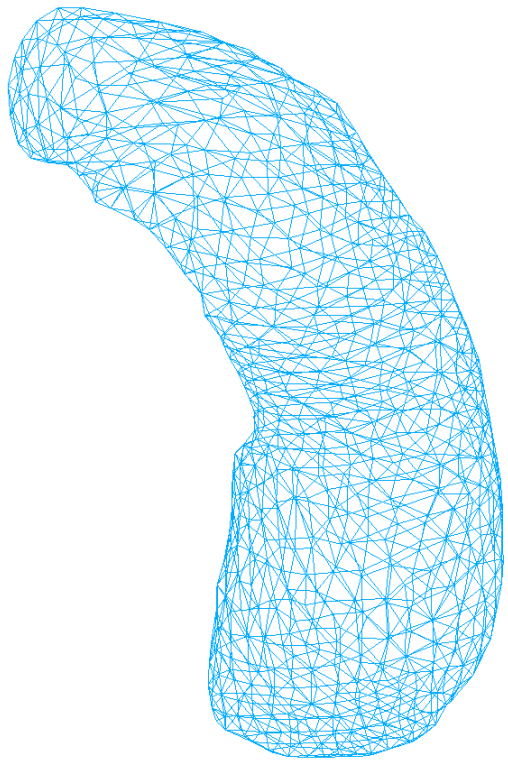
with $(x_1(0), \dots, x_n(0)) = q_0$ and

$$\dot{x}_i(t) = \sum_{j=1}^{n} p_i(t) e^{-\frac{|x_i-x_j(t)|^2}{\sigma^2}}, \quad i = 1, \dots, n.$$

This is a finite dimensional optimal control problem.

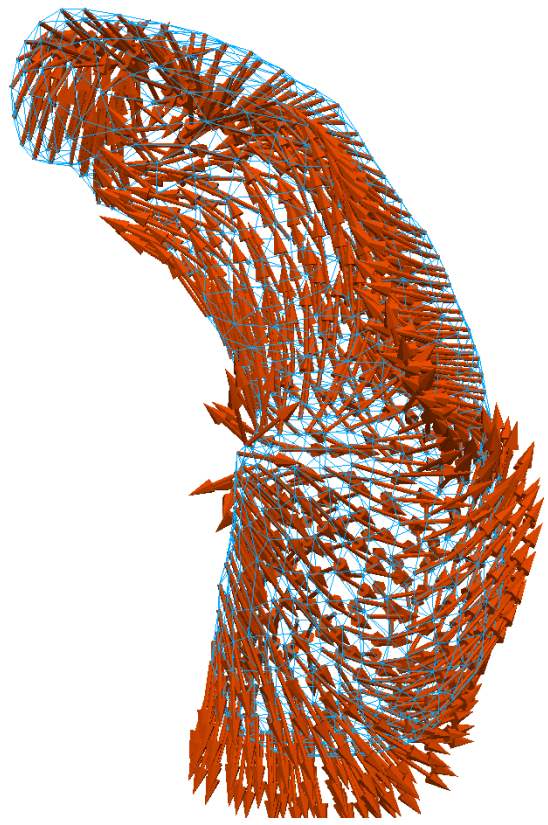
Remark: Choosing $p_i(t)$ yields all of $v^*(t)$: we can still compute the whole diffeomorphism $\varphi(t)$. Useful for finding geometric markers.

Controlling Diffeomorphisms with Vector Fields: Reduced form



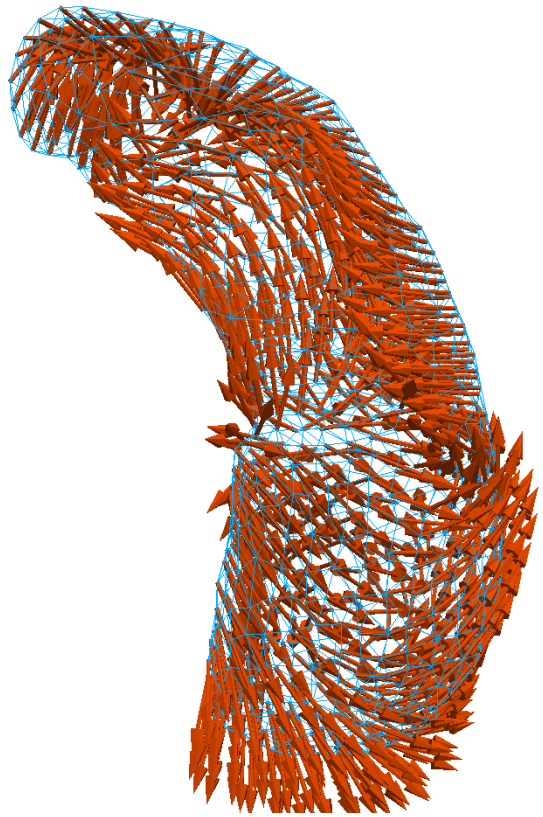
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



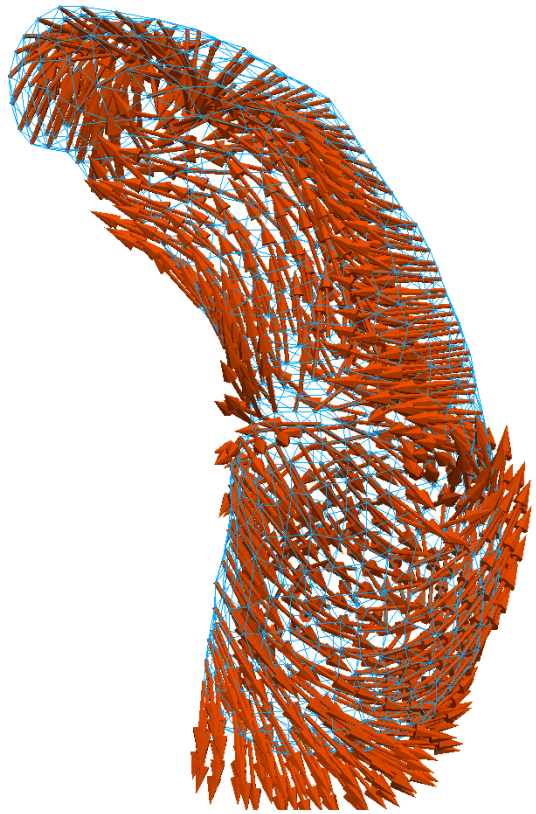
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



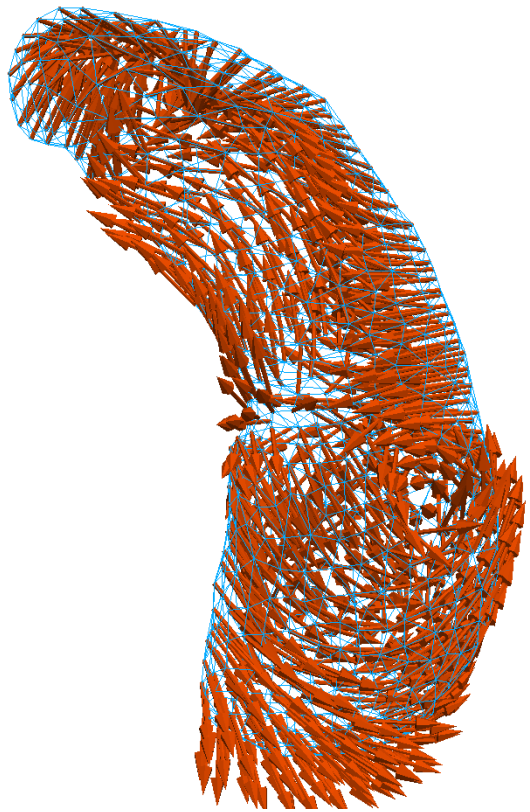
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



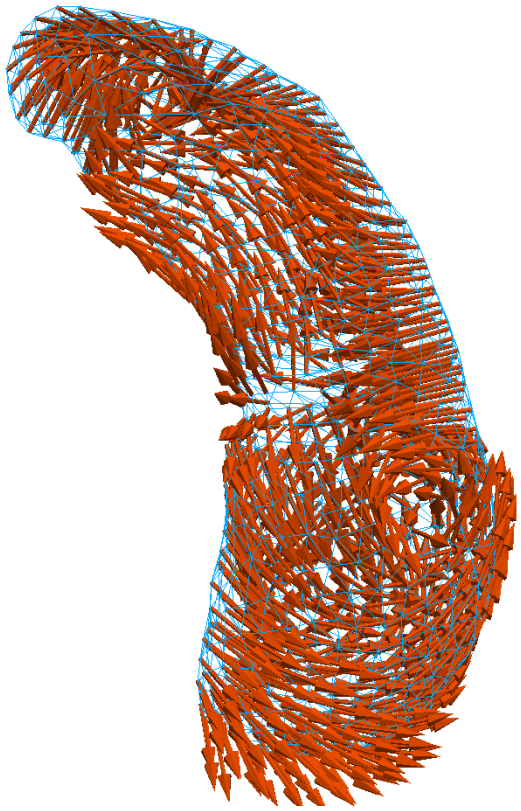
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



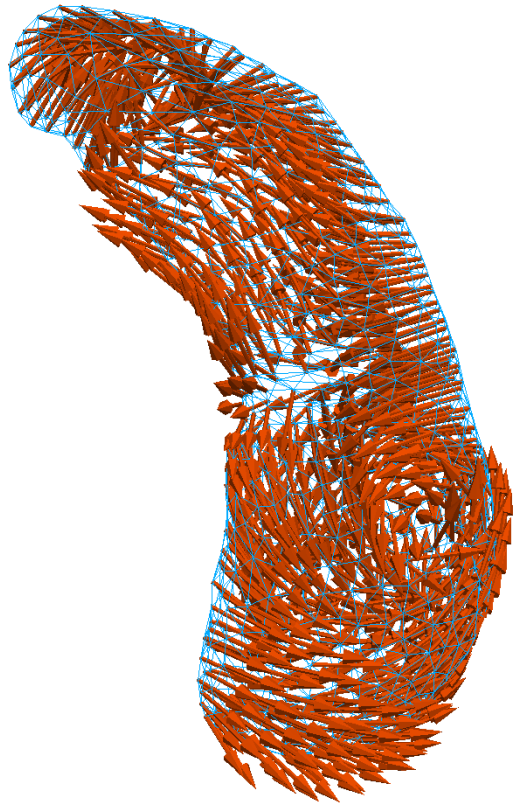
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



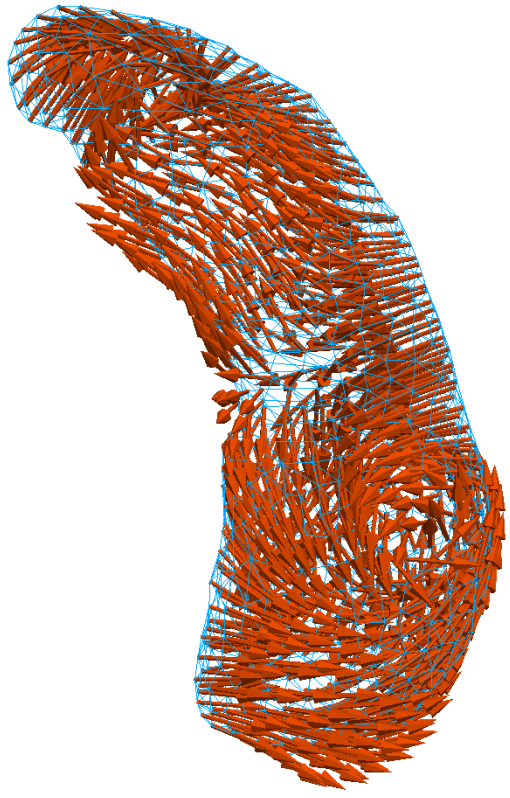
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



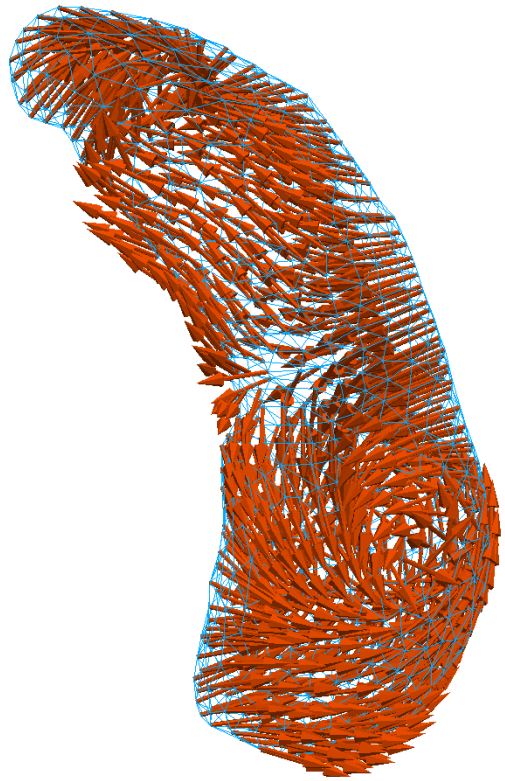
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



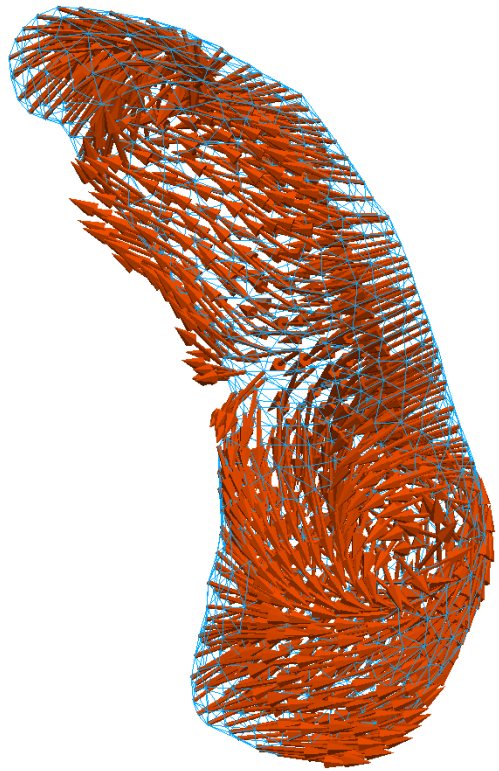
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



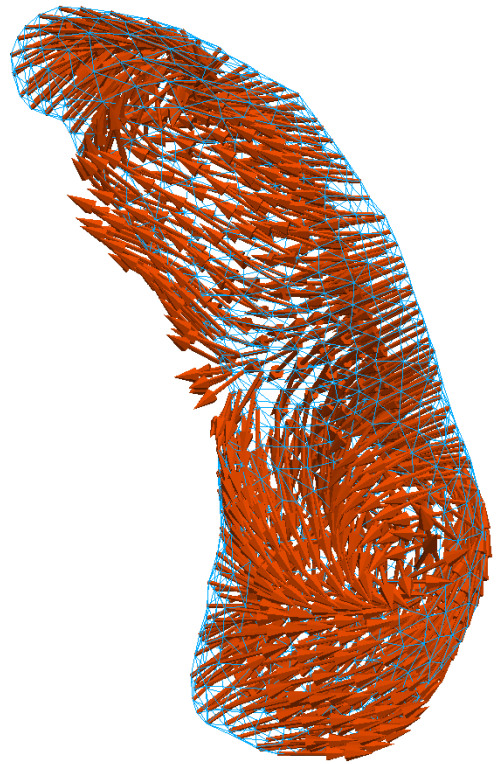
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



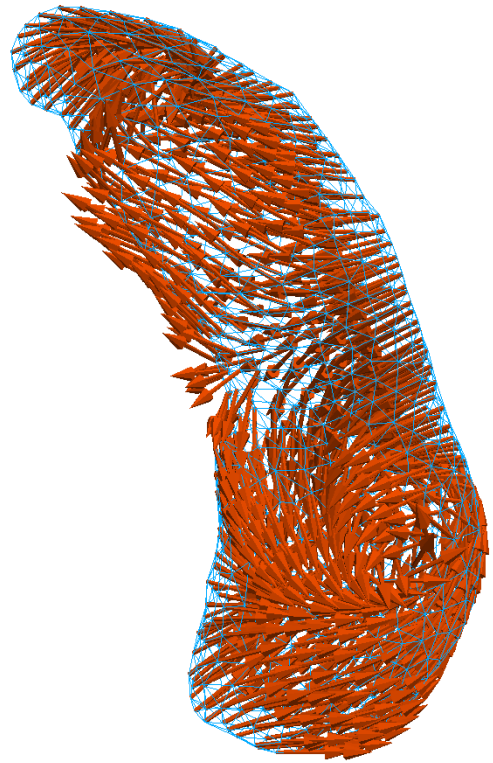
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



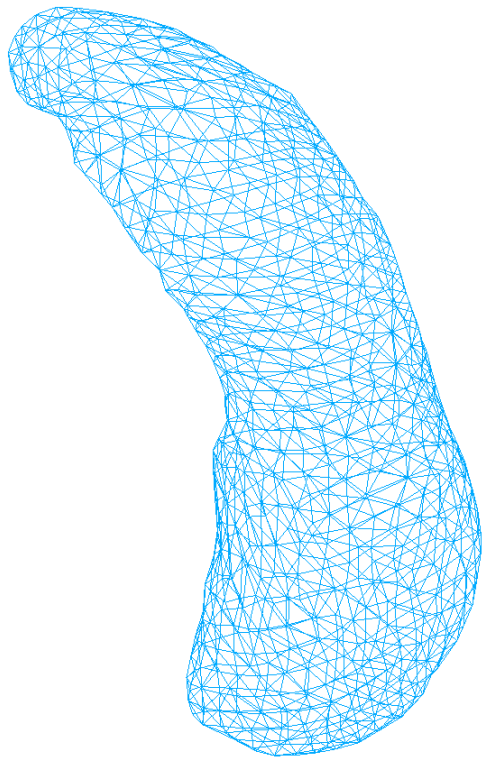
The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



The control is specified at every point of the surface, then interpolated using the kernel to the whole space.

Controlling Diffeomorphisms with Vector Fields: Reduced form



Final Deformed Surface

Pontryagin Principle

Consider the Hamiltonian

$$H(q,p) = H(x_1,\dots,x_n,p_1,\dots,p_n) = \frac{1}{2}\sum_{i,j=1}^{n} p_i(t)^T p_j(t) e^{-\frac{|x_i(t)-x_j(t)|^2}{\sigma^2}}.$$

Theorem

For reasonnable $\|\cdot\|_V$ and g, minimizers $(t,x)\mapsto v(t,x)$ of J are completely determined by the value of $p(0)=(p_1(0),\dots,p_n(0))$, through the Hamiltonian equation

$$\dot{x}_i(t) = v(t,x_i(t)) = \nabla_{p_i} H(q(t),p(t)) = \sum_{j=1}^{n} p_j(t) e^{-\frac{|x_i(t)-x_j(t)|^2}{\sigma^2}},$$

$$\dot{p}_i(t) = -\nabla_{x_i} H(q(t),p(t)).$$

Moreover, in this case,

$$J(v) = H(q_0,p(0)) + g(q(1)).$$

Registration

Registration problem reduced to minimizing

$$\tilde{J}(p_0) = H(q_0, p_0) + g(q(1)),$$

where $q(1)$ is obtained by solving the previous Hamiltonian equation with $q(0) = q_0$ and $p(0) = p_0$. The corresponding minimizing initial momentum $p_0^* = p_0(q_1)$ completely encodes the deformation from q_0 to q_1.

Karcher Mean and Hypertemplate

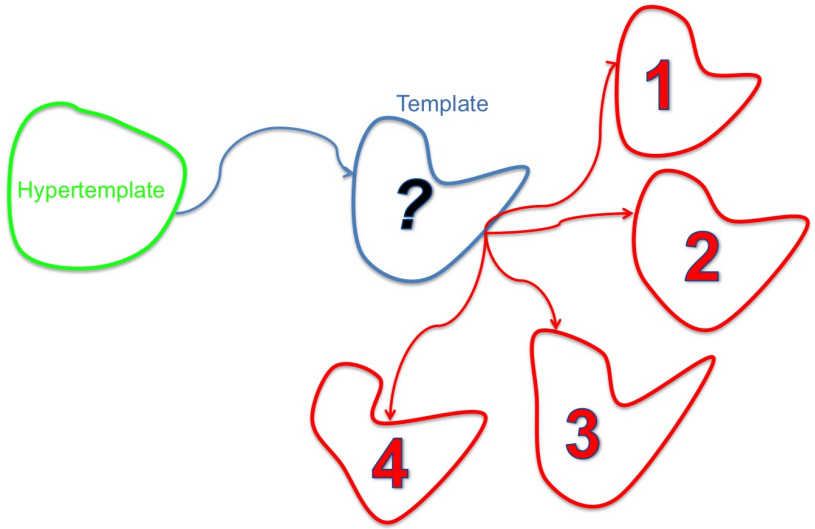
Assume we have $\tilde{q}_1, \ldots, \tilde{q}_k$, k distinct shapes, representing same organ/part of the brain among various patients. First step for statistical analysis: compute average shape, which will be used as the template from which all shapes are registered.

Problem: average unknown, and the $\tilde{q}_i$ are generally noisy/not well adapted to apply diffeomorphisms to.

Karcher Mean and Hypertemplate

Assume we have $\tilde{q}_1, \ldots, \tilde{q}_k$, k distinct shapes, representing same organ/part of the brain among various patients. First step for statistical analysis: compute average shape, which will be used as the template from which all shapes are registered.

Problem: average unknown, and the $\tilde{q}_i$ are generally noisy/not well adapted to apply diffeomorphisms to.

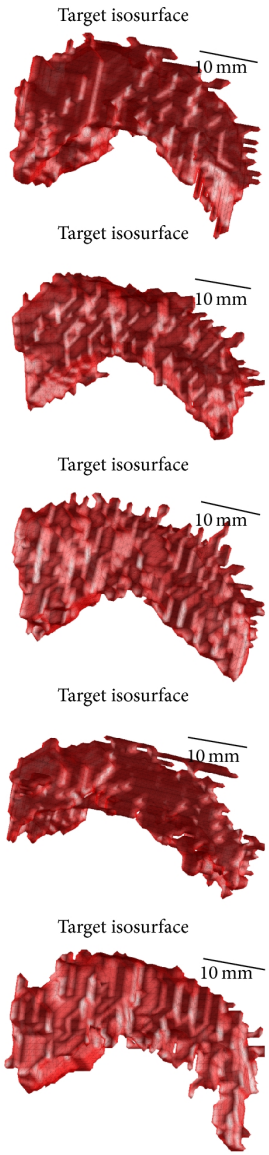


Take nice hypertemplate q_0. Minimize w.r. to $v_0, \tilde{v}_1, \ldots, \tilde{v}_k$ the functional

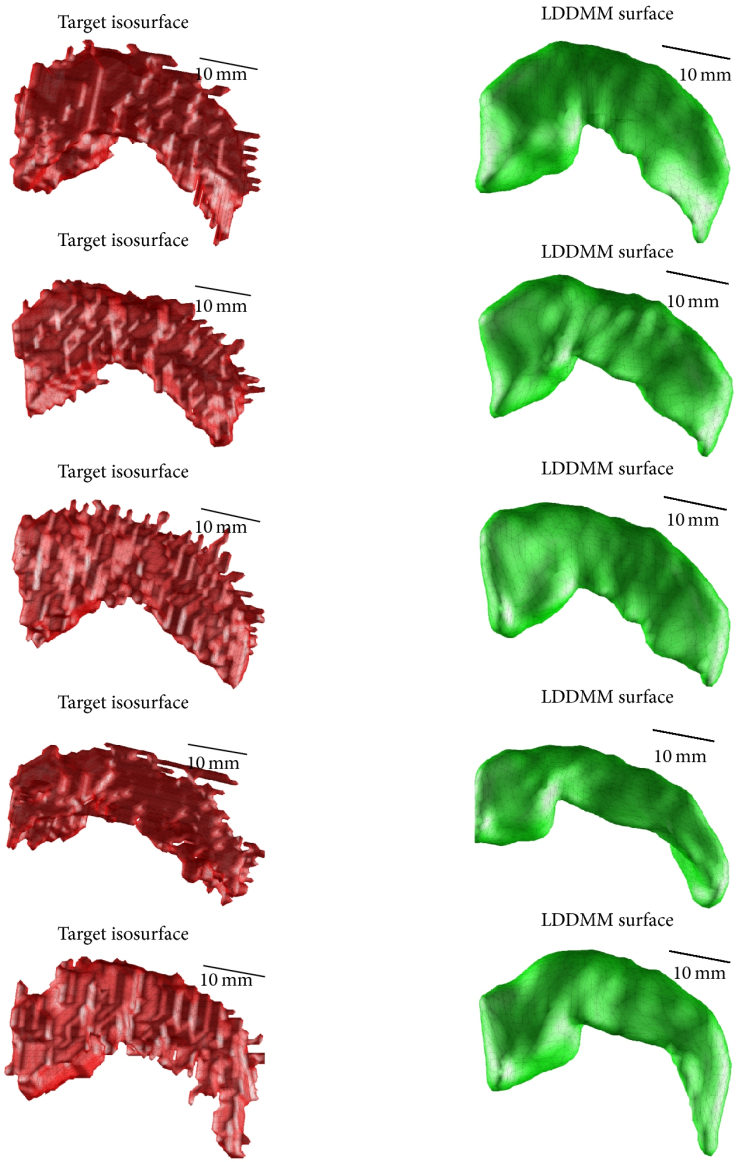
$$\lambda \int_0^1 \|v_0(t)\|^2 dt + \frac{1}{2} \sum_{i=1}^{k} \int_0^1 \|\tilde{v}_i(t)\|^2 dt + g_i(\tilde{\varphi}_i(1) \circ \varphi_0(1) \cdot q_0).$$

The average shape will be $\bar{q} = \varphi_0(1) \cdot q_0$, and we will simultaneously have registered every $\tilde{q}_k$ from $\bar{q}$ through some initial momentum p_k from $\bar{q}$, from which we can deduce corresponding the velocity fields that bring $\bar{q}$ to $\tilde{q}_k$.

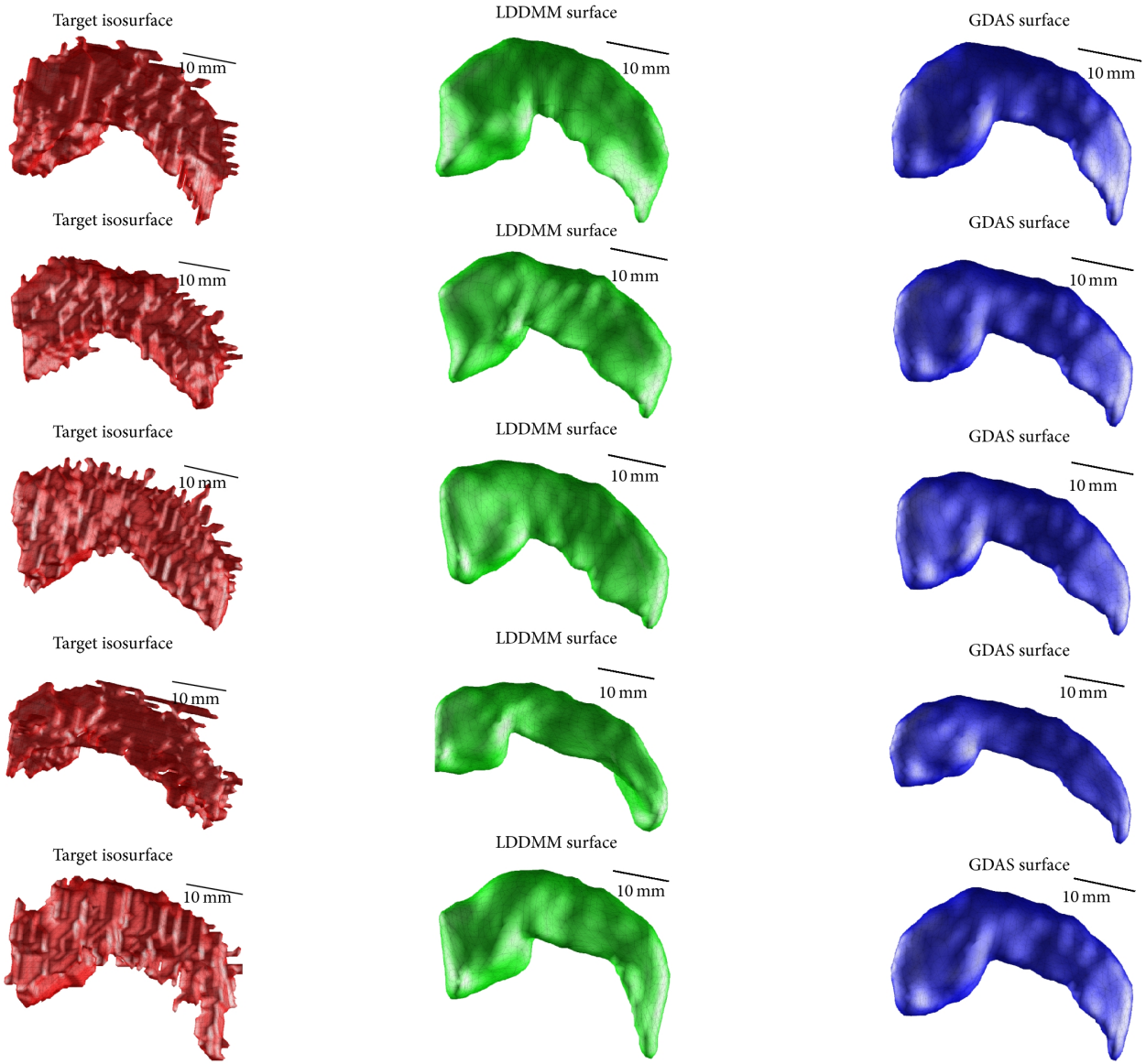
Example of Application: Smoothing Data



Example of Application: Smoothing Data



Example of Application: Smoothing Data



Other Geometric Markers

- We have an average shape $\bar{q}$ over a data set $\tilde{q}_1, \ldots, \tilde{q}_k$, registered as initial momenta $p_1, \ldots, p_k$ along $\bar{q}$. However, it is hard to directly give them a geometric meaning.

- Instead, each p_k yields a corresponding minimizing vector field $\tilde{v}_k(t)$ which integrates into a deformation of the space $\tilde{\varphi}_k(1)$. These have a precise geometric interpretation.

Other Geometric Markers

- We have an average shape $\bar{q}$ over a data set $\tilde{q}_1, \ldots, \tilde{q}_k$, registered as initial momenta $p_1, \ldots, p_k$ along $\bar{q}$. However, it is hard to directly give them a geometric meaning.

- Instead, each p_k yields a corresponding minimizing vector field $\tilde{v}_k(t)$ which integrates into a deformation of the space $\tilde{\varphi}_k(1)$. These have a precise geometric interpretation.

- For example, when studying degenerative diseases, one can compute the change of volume between the average shape $\bar{q}$ and each deformed shape $\tilde{\varphi}_k \cdot \bar{q}$. This can be used to differentiate controls from sick patients in a study.

Other Geometric Markers

- We have an average shape $\bar{q}$ over a data set $\tilde{q}_1, \ldots, \tilde{q}_k$, registered as initial momenta $p_1, \ldots, p_k$ along $\bar{q}$. However, it is hard to directly give them a geometric meaning.
- Instead, each p_k yields a corresponding minimizing vector field $\tilde{v}_k(t)$ which integrates into a deformation of the space $\tilde{\varphi}_k(1)$. These have a precise geometric interpretation.
- For example, when studying degenerative diseases, one can compute the change of volume between the average shape $\bar{q}$ and each deformed shape $\tilde{\varphi}_k \cdot \bar{q}$. This can be used to differentiate controls from sick patients in a study.
- We can be even more precise: compute the (total, surface, normal) jacobian of each deformation $\tilde{\varphi}_k$ at each point of the template, or the elastic strain along certain direction when getting from the template to one of the data points...

Application: BIOCARD Study

By M. Miller, M. Albert, L. Younes et al.

- 1995-2008: Alzheimer's disease longitunal study at NIH
- 350 healthy subjects with large proprotion at risk of dementia and AD
- 1-6 MRI scans per subject
- Goal: Identify shape structures that are primarily affected (ERC, hippocampus, amygdala).

Application: BIOCARD Study

- All subjects were healthy at beginning of study
- At end of study, 66 patients diagnosed with mild cognitive impairment or dementia.
- Longitudinal model comparing differences between controls (healthy until end of study) and MCI patients.

Application: BIOCARD Study

Table 3
Annualized atrophy rates for normal group and preclinical AD group.

Groups	Amygdala mm^3/year	Amygdala %/year	Hippocampus mm^3/year	Hippocampus %/year	ERC mm^3/year	ERC %/year	ERC thickness mm/year	ERC thickness %/year
L controls (n = 81)	4.6 ± 39.1	0.2 ± 2.7	14.0 ± 25.2	0.5 ± 0.9	4.9 ± 19.8	0.8 ± 4.3	.008 ± 0.043	0.34 ± 1.89
L preclinical (n = 20)	16.8 ± 25.3	1.0 ± 1.6	14.4 ± 29.1	0.5 ± 1.1	8.1 ± 15.3	1.7 ± 3.2	.022 ± 0.050	0.92 ± 2.20
L ApoE4+ (n = 44)	4.9 ± 45.3	0.2 ± 3.2	12.9 ± 29.6	0.6 ± 1.0	4.8 ± 20.8	0.7 ± 4.6	.019 ± 0.047	0.76 ± 2.02
L ApoE4− (n = 73)	8.2 ± 32.4	0.4 ± 2.1	11.9 ± 24.3	0.4 ± 0.9	5.1 ± 19.5	0.5 ± 6.5	.004 ± 0.054	−0.01 ± 3.22
R controls (n = 81)	14.2 ± 29.8	0.9 ± 2.0	21.2 ± 31.8	0.9 ± 1.9	5.5 ± 19.7	0.9 ± 4.2	.007 ± 0.039	0.28 ± 1.78
R preclinical (n = 20)	22.0 ± 27.1	1.4 ± 1.8	4.6 ± 28.5	1.1 ± 1.4	13.2 ± 19.2	3.3 ± 3.8	.024 ± 0.040	1.08 ± 1.79
R ApoE4+ (n = 44)	14.5 ± 28.6	1.0 ± 2.0	20.8 ± 33.4	0.8 ± 1.8	8.4 ± 24.1	1.6 ± 4.9	.014 ± 0.047	0.60 ± 2.10
R ApoE4− (n = 73)	16.7 ± 37.1	1.0 ± 2.4	14.5 ± 30.4	0.9 ± 2.3	4.3 ± 17.3	0.6 ± 4.7	.001 ± 0.040	−0.03 ± 2.18
B controls (n = 81)	9.4 ± 27.6	0.6 ± 1.8	17.6 ± 22.4	0.7 ± 0.9	5.2 ± 14.8	1.0 ± 3.3	.008 ± 0.031	0.33 ± 1.42
B preclinical (n = 20)	19.4 ± 19.2	1.2 ± 1.2	9.5 ± 20.6	0.3 ± 0.8	10.6 ± 14.4	2.7 ± 3.1	.023 ± 0.039	1.04 ± 1.73
B ApoE4+ (n = 44)	9.7 ± 29.4	0.6 ± 2.0	16.9 ± 25.1	0.7 ± 0.9	6.6 ± 16.2	1.3 ± 3.4	.016 ± 0.034	0.71 ± 1.48
B ApoE4− (n = 73)	12.5 ± 30.0	0.7 ± 1.9	13.2 ± 20.7	0.5 ± 0.8	4.7 ± 15.9	0.6 ± 5.5	.002 ± 0.042	0.01 ± 2.47

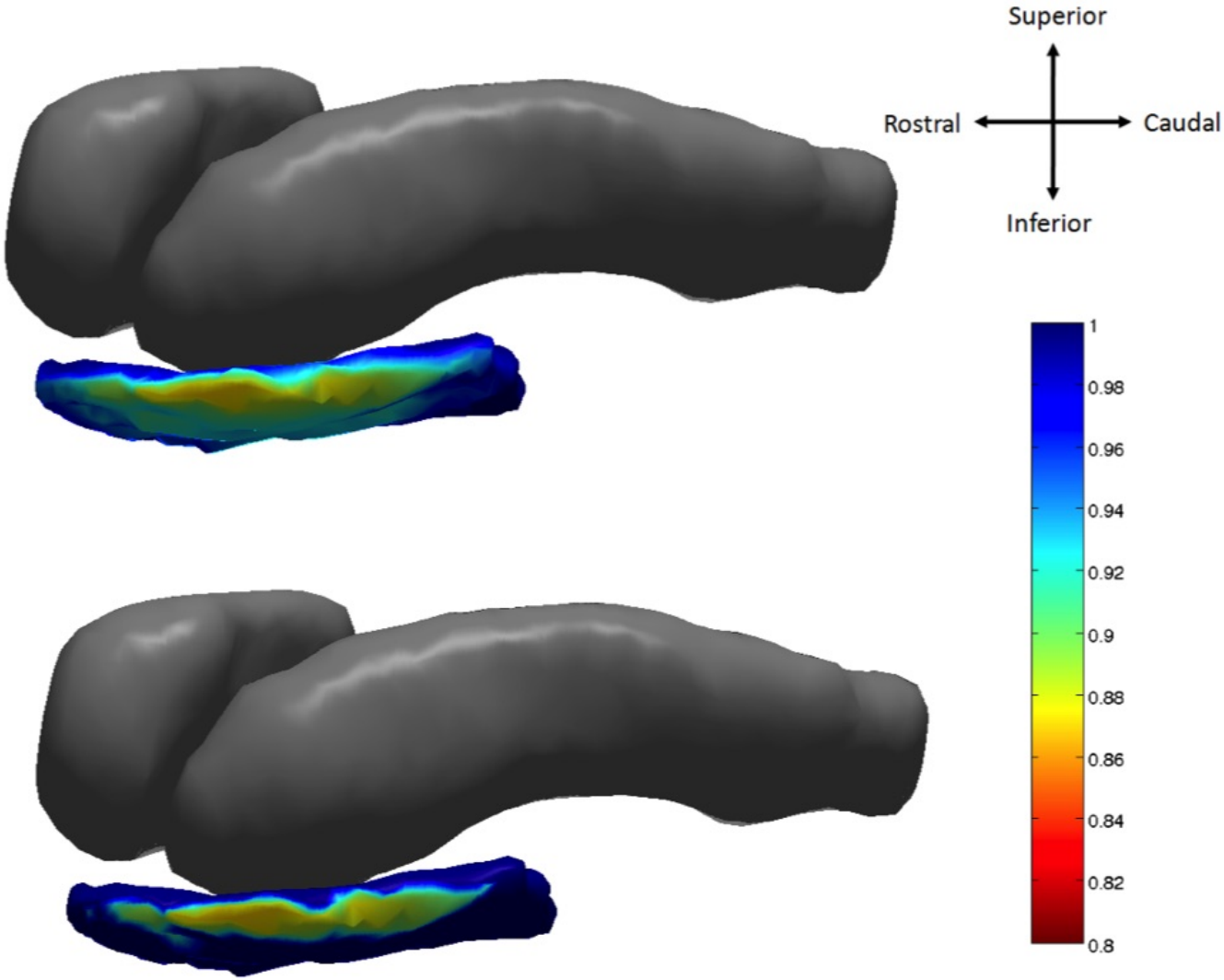
The table presents the volume atrophy rates and standard deviations in % and mm^3/year for amygdala (columns 2 and 3), hippocampus (columns 4 and 5) and entorhinal cortex (ERC) (columns 6 and 7), for time series with at least 3 scans. The top group of four rows is for L = Left; the middle group of four rows is for R = Right; the bottom group of four rows is for B = Bilateral; three preclinical subjects with hippocampal volume atrophy rates were outliers and were removed.

Application: BIOCARD Study

Morphometry measures comparing normal group vs preclinical AD group.

Structures examined	p-Values based on vertex measure Control vs. preclinical AD	p-Values based on Laplace measure Controls vs. preclinical AD	p-Values based on volume measure Controls vs. preclinical AD
Amygdala (L)	0.17	0.13	0.0086
Hippocampus (L)	0.022	0.33	0.073
ERC (L)	<0.0001	0.0001	0.51
Amygdala (R)	0.031	0.029	0.0043
Hippocampus (R)	0.0025	0.08	0.79
ERC (R)	0.0067	0.0003	0.17

Application: BIOCARD Study



Going further:

Many variants and generalizations:

- Using sums of kernels to study various scales of deformations
- Many possibilities for g, usually currents (Glaunès) or varifolds (Charon)
- Deformation modules (Gris): flexible generalization with "explicit" constraints on the deformations, also allows the learning of those constraints/types of deformations/metrics.

For following time-varying shapes, it would be good to take physical properties of the objects studied into accounts.

- Geometric Control viewpoint (A., Azencott, Gris, Trélat, Trouvé, Younes...) allows flexible addition of constraints on the deformation such as imposing volume-decreasing diffeomorphisms.
- Growth simulation (Kaltenmark)
- One can also add various elastic energies on the deformations (ongoing work with Charon, Hsieh, Younes).

Thank you for your attention!